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The seven classes of 5×6 triple arrays

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Abstract

Triple arrays are considered in which 10 symbols each appear 3 times in a 5×6 arrangement of symbols. These triple arrays fall into seven isomorphism classes. The orders of the automorphism groups of the arrays in these classes are 60, 12, 12, 6, 4, 3 and 3, respectively.
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1. Definitions

As defined in [3], a *triple array* is an $r \times c$ row–column arrangement of elements taken from a set of v symbols (which we represent as $1, 2, \ldots, v$), such that

(i) each symbol occurs exactly k times in the array, so that the constant k satisfies $k|rc$;
(ii) any two distinct rows have $c(k-1)/(r-1)$ elements in common;
(iii) any two distinct columns have $r(k-1)/(c-1)$ elements in common;
(iv) any row and any column have k elements in common.

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An example of a 5×6 triple array with $v=10$ and $k=3$ is the following, where columns and rows are labelled, for future reference, with upper case and lower case letters, respectively:

	A	B	C	D	E	F
a	6	4	10	7	3	1
b	4	7	5	6	8	2
c	9	5	8	1	7	3
d	8	10	1	9	2	4
e	3	9	6	2	10	5

An early example of a 5×6 triple array with $v = 10$ and $k = 3$ was given in the statistical literature by Agrawal [1, p. 1157].

From the above definition, a 5×6 triple array must have $k = 1$, 3 or 5. The first of these possibilities implies that $v = 30$, so that all 30 entries are distinct, whereas $k = 5$ implies that the array is a 6×6 Latin square with a row missing. We ignore these two extreme possibilities, and henceforth require a 5×6 triple array to have $k = 3$.

We say that a 5×6 triple array is *standardised* if the symbols $1, 2, \ldots, 6$ appear in that order in row 1, if the symbols 7, 8, 9 appear in that order in the first three positions in row 2, and if the symbols in the last three positions in column 1 are in numerical order.

We say that two 5×6 triple arrays are *isomorphic* to one another if one can be obtained from the other by some combination of the operations

(a) a permutation of the rows,
(b) a permutation of the columns and
(c) a permutation of the symbols $1, 2, \ldots, v$.

If a 5×6 triple array is invariant under such a combination of operations, then the permutations together constitute an *automorphism* of the array. As usual, the complete set of automorphisms of any particular triple array constitutes the *automorphism group* of that array.

Let $\mathbf{n}_\mathrm{C}$ be the 6×10 matrix whose (i, j)th entry is 1 if symbol j appears in column i, or is 0 otherwise ($i = 1, 2, \ldots, 6$; $j = 1, 2, \ldots, 10$). Likewise let $\mathbf{n}_\mathrm{R}$ be the 5×10 matrix whose (i, j)th entry is 1 if symbol j appears in row i, or is 0 otherwise ($i = 1, 2, \ldots, 5$; $j = 1, 2, \ldots, 10$). Then $\mathbf{n}_\mathrm{C}$ is the incidence matrix of a balanced incomplete block design $\mathscr{D}_\mathrm{C}$ with 6 treatments each replicated 5 times in 10 blocks each of size 3, and $\mathbf{n}_\mathrm{R}$ is the incidence matrix of a balanced incomplete block design $\mathscr{D}_\mathrm{R}$ with 5 treatments each replicated 6 times in 10 blocks each of size 3.

We define the *structural graph* of a 5×6 triple array to be the graph in which

(i) there is one node for each of the 5 rows, one node for each of the 6 columns, and one node for each of the 10 symbols, so that there are 21 nodes in total;
(ii) for each symbol there are edges joining its node to the nodes for the 3 rows and the 3 columns in which that symbol appears, so that there are 60 edges in total.

It follows from our definitions that the automorphism group of the structural graph of a 5×6 triple array is a subgroup (not necessarily proper) of the automorphism group of $\mathscr{D}_\mathrm{C}$ and of the automorphism group of $\mathscr{D}_\mathrm{R}$. Likewise the automorphism group of a particular

5×6 triple array is a subgroup (not necessarily proper) of the automorphism group of the triple array's structural graph.

2. Enumerating 5×6 triple arrays

Given any 5×6 triple array, we can re-order its columns so that the first three elements to appear in the second row of the array are absent from the first row. We can then re-order the third, fourth and fifth rows so that column 1 of the array has its entries in these rows in numerical order. Accordingly, we can obtain all the 5×6 triple arrays by first generating all standardised 5×6 triple arrays.

Straightforward computer enumeration showed that there are 912 standardised triple arrays. The computer package `nauty` [2] was used to test these for isomorphism and to obtain the automorphism groups of non-isomorphic 5×6 triple arrays. This showed that the 5×6 triple arrays fall into 7 classes as in Table 1, where A_n, C_n and S_n denote, respectively, the alternating, cyclic and symmetric groups on n elements. (The order of S_n is $n!$, and the order of A_n is $n!/2$.) The product of the number of standardised designs and the order of the automorphism group is constant from one class to another.

Theorem 5.2 of [3] shows that the design $\mathscr{D}_C$ must be the residual design of a symmetric balanced incomplete block design $\mathscr{S}$ with $(v, k, \lambda) = (11, 5, 2)$ with regard to some block of $\mathscr{S}$, and $\mathscr{D}_R$ must be the complement of the derived design with regard to the same block. Up to isomorphism, there is only one symmetric balanced incomplete block design with $(v, k, \lambda) = (11, 5, 2)$, and up to isomorphism, this symmetric balanced incomplete block design has only one residual design to provide us with $\mathscr{D}_C$, and only one derived design to provide us with $\mathscr{D}_R$. It follows therefore that, up to isomorphism, there is only one possible structural graph for a 5×6 triple array. This structural graph has automorphism group isomorphic to A_5, and Table 1 is of course consistent with this result. The automorphism groups of $\mathscr{D}_C$ and $\mathscr{D}_R$ are isomorphic, respectively to A_5 and S_5, with which the structural graph is of course consistent.

The 5×6 triple array in Section 1 is 5-cyclic, clearly being invariant under the set of permutations (ABCDE)(abcde)(1 2 3 4 5)(6 7 8 9 10). That particular triple array must therefore

Table 1
Classification of 5×6 triple arrays

Class	No. of standardised triple arrays	Order of automorphism group	Automorphism group
1	12	60	A_5
2	60	12	A_4
3	60	12	A_4
4	120	6	S_3
5	180	4	$C_2 \times C_2$
6	240	3	C_3
7	240	3	C_3
Total	912		

Table 2
Specimen members of the 7 classes of 5×6 triple arrays

Class 1:

	B	C	F	D	A	E
b	7	5	2	6	4	8
e	9	6	5	2	3	10
c	5	8	3	1	9	7
a	4	10	1	7	6	3
d	10	1	4	9	8	2

Automorphism group A_5
generators:
(1 2 3 4 5)(6 7 8 9 10)(ABCDE)(abcde)
(1 4 10)(2 9 8)(3 7 6)(BCF)(DAE)(bec)

Class 2:

	B	C	F	D	A	E
b	5	6	2	7	4	8
e	9	5	3	2	6	10
c	7	8	5	1	9	3
a	4	10	1	6	3	7
d†	10	1	4	9	8	2

Automorphism group A_4
generators:
(1 4 10)(2 9 8)(3 7 6)(BCF)(DAE)(bec)
(1 2)(3 6)(4 9)(5 7)(CE)(FD)(bc)(ea)

Class 3:

	B	C	F	D	A	E
b	4	8	5	2	6	7
e	5	10	2	6	9	3
c	9	5	1	7	3	8
a	7	6	3	1	4	10
d†	10	1	4	9	8	2

Automorphism group A_4
generators:
(1 4 10)(2 9 8)(3 7 6)(BCF)(DAE)(bec)
(1 2)(3 5)(6 7)(8 10)(BA)(CE)(ba)(ec)

Class 4:

	B	C	F	D	A	E
b	7	5†	2	6	4	8
e	9	6	5†	2	3	10
c	5†	8	3	1	9	7
a	10	1	4	7	6	3
d	4	10	1	9	8	2

Automorphism group S_3
generators:
(1 4 10)(2 9 8)(3 7 6)(BCF)(DAE)(bec)
(1 4)(2 3)(6 9)(7 8)(BC)(DA)(bc)(ad)

Class 5:

	B	C	F	D	A	E
b	5	8	2	6	4	7
e	9	10	5	2	6	3
c	7	5	3	1	9	8
a	4	6	1	7	3	10
d†	10	1	4	9	8	2

Automorphism group $C_2 \times C_2$
generators:
(1 2)(3 6)(4 9)(5 7)(CE)(FD)(bc)(ea)
(1 2)(4 9)(5 6)(8 10)(BA)(FD)(be)(ca)

Class 6:

	B	C	F	D	A	E
b	5†	6	2	7	4	8
e	9	5†	3	2	6	10
c	7	8	5†	1	9	3
a†	10	1	4	6	3	7
d†	4	10	1	9	8	2

Automorphism group C_3
generator:
(1 4 10)(2 9 8)(3 7 6)(BCF)(DAE)(bec)

Table 2 (*continued*)

Class 7:

	B	C	F	D	A	E
b	$5^{\dagger}$	8	4	2	6	7
e	10	$5^{\dagger}$	2	6	9	3
c	9	1	$5^{\dagger}$	7	3	8
$a^{\dagger}$	7	6	3	1	4	10
$d^{\dagger}$	4	10	1	9	8	2

Automorphism group C_3
generator:
(1 4 10)(2 9 8)(3 7 6)(BCF)(DAE)(bec)

belong to Class 1 from Table 1. It is, however, also invariant under (BCF)(DAE)(bec)(1 4 10) (2 9 8)(3 7 6), as can readily be seen by re-ordering the rows and columns to give the triple array in the following form:

	B	C	F	D	A	E
b	7	5	2	6	4	8
e	9	6	5	2	3	10
c	5	8	3	1	9	7
a	4	10	1	7	6	3
d	10	1	4	9	8	2

We use this re-ordering as the basis for our Table 2, which contains a member of each of the seven classes of 5×6 triple arrays. We judge that the distinctions between the classes are thereby most clearly seen, the notion of standardised 5×6 triple array having been useful only for the programming of the enumeration. The first triple array in Table 2 is the 5×6 triple array just given.

Throughout Table 2, the different triple arrays have the same symbols in corresponding columns and in corresponding rows. The arrays for Classes 2 and 3, which have the same automorphism group, are readily distinguishable, as the former has a 2×3 Latin rectangle in the last two rows whereas the latter does not. The same is true of the arrays for Classes 6 and 7.

A triple array from Class 1 has a 2×3 Latin rectangle embedded in each pair of its rows, so the array contains 10 2×3 Latin rectangles in total. Triple arrays from Classes 2 and 4 contain four 2×3 Latin rectangles each, whereas those from Classes 3 and 7 contain none. Triple arrays from Class 5 have two 2×3 Latin rectangles each, and those from Class 6 have one.

Some of the 5×6 triple arrays have rows that are fixed under all automorphisms; each such row is indicated in Table 2 by a dagger(†). Likewise a fixed symbol in a triple array in Table 2 is marked with a dagger.

3. Triple arrays as Graeco–Latin designs

Every $r \times c$ triple array is equivalent to a special type of non-orthogonal Graeco–Latin block design with v blocks, each of size k, wherein a set of r treatments is superimposed on a set of c treatments. The properties of a triple array are such that the statistical efficiency for either of these two sets of treatments, in the presence of the other set, is the same as if the other set were absent, on the assumption that treatments from different sets do not interact with one another (see [4]).

Thus the 5×6 triple array from Section 1,which is from our present Class 1, is equivalent to the following Graeco–Latin block design, where the block labels $1, 2, \ldots, 10$ correspond exactly to the previous symbols $1, 2, \ldots, 10$, and where the three treatment-pairs in block i ($i = 1, 2, \ldots, 10$) are the three column-row pairs for symbol i in the rectangular form of the triple array:

1	(Fa	Dc	Cd)	6	(Aa	Db	Ce)
2	(Fb	Ed	De)	7	(Bb	Ec	Da)
3	(Fc	Ae	Ea)	8	(Cc	Ad	Eb)
4	(Fd	Ba	Ab)	9	(Dd	Be	Ac)
5	(Fe	Cb	Bc)	10	(Ee	Ca	Bd)

This Graeco–Latin block design, being 5-cyclic, can be represented concisely by its initial blocks 1 and 6: { (Fa Dc Cd) (Aa Db Ce) }. The design was given in this form in Table 1 of Preece [4, pp. 12–13]. Now, a full choice of Graeco–Latin block designs with the same basic properties and parameters is available. All 5-cyclic examples belong to Class 1, but there are six having Fa in the first initial block and Aa in the second:

{ (Fa Dc Cd) (Aa Db Ce) }, { (Fa Ac Ed) (Aa Db Bc) },
{ (Fa Bc Ad) (Aa Ed Ce) }, { (Fa Be Eb) (Aa Bc Ed) },
{ (Fa Ce Ab) (Aa Db Ed) }, { (Fa Ae Db) (Aa Bc Ce) }.

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